A TEST OF HOMOGENEITY OF MEANS VERSUS ALTERNATIVES SATISFYING MONOTONE TREND AND SLOPE

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Abstract

A test of the hypothesis H_o: $\mu(x_1) = \mu(x_2) = \cdots = \mu(x_k)$ is derived for alternatives

$$H_a: \mu(x_1) \leq \mu(x_2) \leq \cdots \leq \mu(x_k) \text{ and } \frac{\mu(x_2) - \mu(x_1)}{x_2 - x_1} \geq \cdots \geq \frac{\mu(x_k) - \mu(x_{k-1})}{x_k - x_{k-1}}$$

which are monotone with monotone slope. The test is based on a contrast of the means, which is chosen to maximize the power of the test for the least favorable configuration of means, and is called the maximin contrast. The test is compared to tests derived for monotone alternatives without regard to slope or the spacing of the treatments (x_i's). An example from a dose-response experiment is given to demonstrate the application of the test.

I. INTRODUCTION

Various tests have been constructed to test H_o: $\mu_1 = \mu_2 = \cdots = \mu_k$ versus H_m: $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_k$ where at least one inequality is strict. [Bartholomew 1959, 1961; Abelson, Tukey 1963; etc.] For many experimental situations however, even

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the monotone alternative may not utilize all of the available prior information, particularly when the treatments are quantitative and the model $E\{Y_i\} = \mu(x_i)$ is applicable where $\mu(x_i)$ is the unknown response function for treatments x_i. Qualitative information in addition to monotonicity can be utilized in specifying restrictions on $\mu(x)$. A restriction on $\mu(x)$, applicable to many experimental situations, is that $\mu(x)$ is monotone with monotone slope. For many dose-response and growth curves, monotone with monotone slope is a reasonable characterization in the dose or treatment range of interest. See Figure 1.

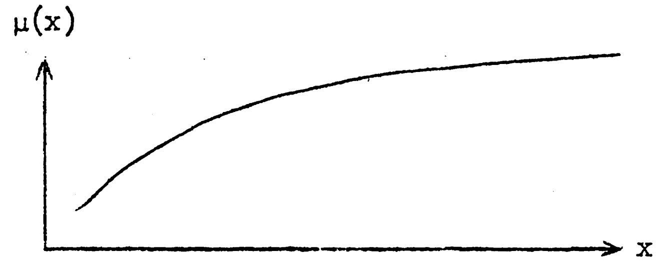


Figure 1.

The tests derived in this report are the most stringent among somewhere most powerful tests for testing H_o: $\mu(x_1) = \mu(x_2) = \cdots = \mu(x_k)$ against alternatives, $\mu(x_i)$, $i = 1, \cdots, k$, in the class

$$H_a: \frac{\mu(x_2) - \mu(x_1)}{x_2 - x_1} \geq \cdots \geq \frac{\mu(x_k) - \mu(x_{k-1})}{x_k - x_{k-1}} \geq 0.$$

Hence we are extending the tests against trend to utilize the information about the quantitative treatments and thus obtain tests with higher minimum power.

II. DESCRIPTION OF THE EXPERIMENTAL SETTING

Let $Y = \mu(x) + \epsilon$ be a response function where Y is the measured response, $\mu(x)$ is a function of the treatment level x such that $\mu(x)$ is monotone with monotone slope, and ϵ is random error distributed as $N(0,\sigma^2)$.

We can consider four cases: (a) $\mu(x)$ monotone increasing with monotone decreasing slope ($\mu(x)\uparrow$, $\mu(x)'\downarrow$); (b) $\mu(x)$ monotone decreasing with monotone decreasing slope ($\mu(x)\downarrow$, $\mu(x)'\downarrow$); (c) $\mu(x)$ monotone decreasing with monotone increasing slope ($\mu(x)\downarrow$, $\mu(x)'\uparrow$); and (d) $\mu(x)$ monotone increasing with monotone increasing slope ($\mu(x)\uparrow$, $\mu(x)'\uparrow$). See Figure 2.a.-d. for schematic displays of these four cases of response functions. Unless otherwise stated, $\mu(x)$ is con-

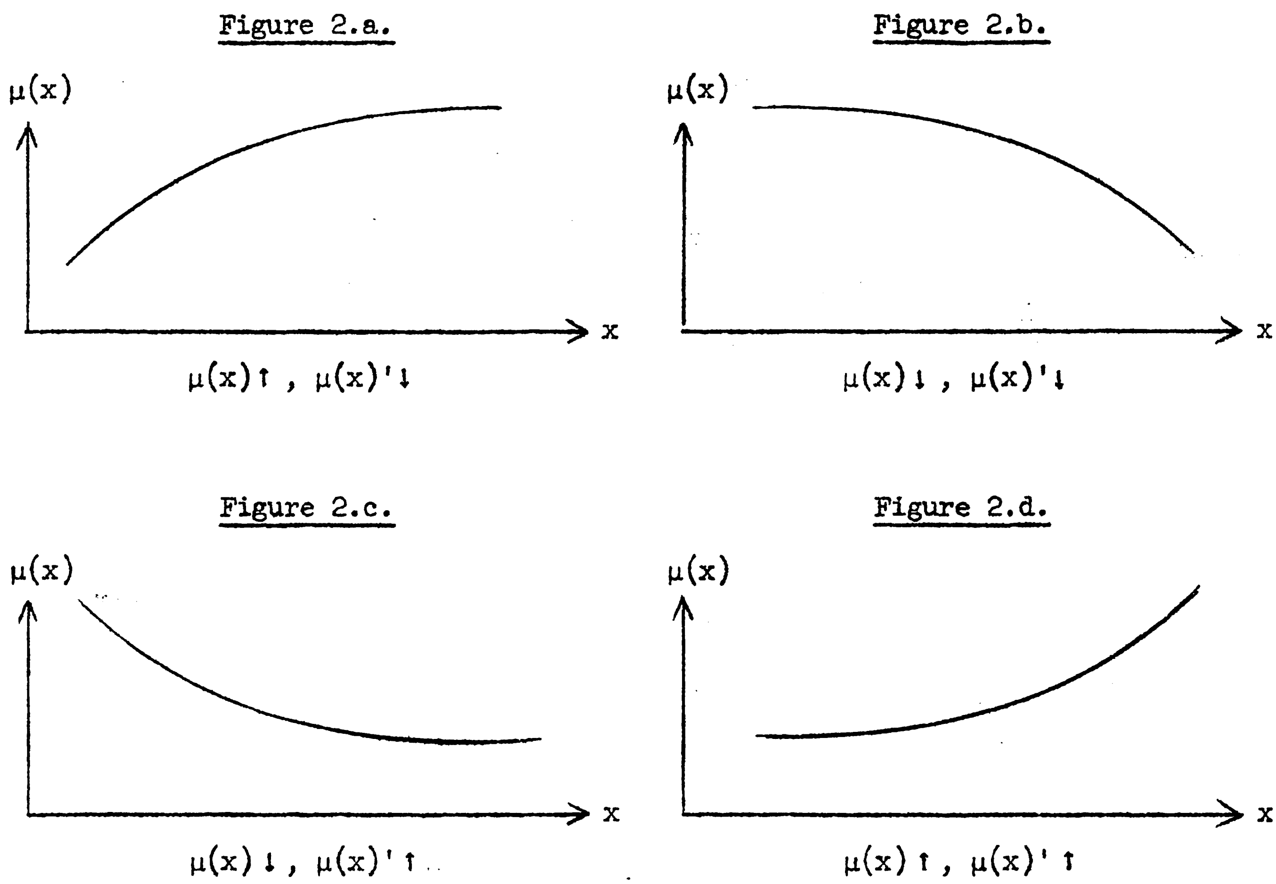


Figure 2: Four cases of monotone response functions with monotone slope.

sidered as monotone increasing with monotone decreasing slope seen in Figure 2.a. and these alternatives are denoted H_a or H_{mc} for monotone concave. The other cases will be considered in Section IV.

A treatment design consists of the selection of k treatment levels or doses, $\{x_1, x_2, \cdots, x_k\}$ which are assumed quantitative with $x_1 < x_2 < \cdots < x_k$. At each treatment level x_i, n observations are taken and the data are denoted Y_{ij}, where $j = 1, \cdots, n$, $i = 1, \cdots, k$. The treatment means are denoted

$$\bar{Y}_{i.} \equiv \frac{1}{n} \sum_{j=1}^{n} Y_{ij}$$

and hence $\bar{Y}_{i.}$ are distributed independently as $N\left(\mu(x_i), \sigma^2/n\right)$.

This experimental setting corresponds to several commonly encountered analysis of variance models. The $\bar{Y}_{i.}$ could be the treatment means of a 1-way analysis of variance in a completely randomized design (CRD) with equal replication of treatments. The estimator of σ^2 would be

$$s^2 \equiv \frac{\sum_{i=1}^{k} \sum_{j=1}^{n} (Y_{ij} - \bar{Y}_{i.})^2}{k(n - 1)}$$

with $\nu \equiv k(n - 1)$ degrees of freedom. The $\bar{Y}_{i.}$ could also be the treatment means of a randomized complete block design (RCBD) with k treatments and n blocks. The estimator of σ^2 would then be

$$s^2 \equiv \frac{\sum_{i=1}^{k} \sum_{j=1}^{n} (Y_{ij} - \bar{Y}_{i.} - \bar{Y}_{.j} + \bar{Y}_{..})^2}{(k - 1)(n - 1)}$$

with $\nu \equiv (k - 1)(n - 1)$ degrees of freedom. Other experimental designs which yield k treatment means with n observations per treatment and an independent estimate of σ^2 could also be utilized.

III. DESCRIPTION OF THE TEST

For a given experimental design with treatment levels $\{x_1, \cdots, x_k\}$ we describe a test of

$$H_o: \mu(x_1) = \mu(x_2) = \cdots = \mu(x_k) \qquad \text{versus}$$

$$H_a: \mu(x_1) \le \mu(x_2) \le \cdots \le \mu(x_k) \;\&\; \frac{\mu(x_2) - \mu(x_1)}{x_2 - x_1} \ge \cdots \ge \frac{\mu(x_k) - \mu(x_{k-1})}{x_k - x_{k-1}}$$

(with at least one strict inequality) which corresponds to a non-trivial response function $\mu(x) \in \{\mu(x) | \mu(x)\uparrow\ ,\ \mu(x)'\downarrow\}$. H_a can be written equivalently and more concisely as

$$H_a: \frac{\mu(x_2) - \mu(x_1)}{x_2 - x_1} \ge \cdots \ge \frac{\mu(x_k) - \mu(x_{k-1})}{x_k - x_{k-1}} \ge 0.$$

The test is based on a contrast of the means, $\sum_{i=1}^{k} c_i\bar{Y}_{i.}$, where $\sum_{i=1}^{k} c_i = 0$. H_o is rejected for large values of $\sum_{i=1}^{k} c_i\bar{Y}_{i.}$. The contrast $\underset{\sim}{c} = (c_1, c_2, \cdots, c_k)$ is chosen to maximize the power of the test against the least favorable configuration of the $\underline{\mu(x)} = \left(\mu(x_1), \mu(x_2), \cdots, \mu(x_k)\right)$ in H_a.

Since $\bar{Y}_{i.} \sim N(\mu(x_i), \sigma^2/n)$ and independent,

$$\sum_{i=1}^{k} c_i\bar{Y}_{i.} \sim N\left(\sum_{i=1}^{k} c_i\mu(x_i), \frac{\sigma^2}{n} \sum_{i=1}^{k} c_i^2\right) \quad \text{and} \quad \frac{\sum_{i=1}^{k} c_i\bar{Y}_{i.} - \sum_{i=1}^{k} c_i\mu(x_i)}{\sigma\sqrt{\sum_{i=1}^{k} c_i^2/n}} \sim N(0,1).$$

For σ^2 unknown, the usual experimental situation, s^2 with ν degrees of freedom will be used as an estimator of σ^2 and hence

$$\frac{\sum_{i=1}^{k} c_i\left(\bar{Y}_{i.} - \mu(x_i)\right)}{s\sqrt{\sum_{i=1}^{k} c_i^2/n}} \sim t_\nu,$$

a t distribution with ν degrees of freedom. Thus the α-level Neyman-Pearson test is defined by:

$$\text{Reject } H_o \text{ if } \frac{\sum_{i=1}^{k} c_i\bar{Y}_{i.}}{s\sqrt{\sum_{i=1}^{k} c_i^2/n}} > t_\nu(1 - \alpha) \qquad (1)$$

where $P\left\{T \leq t_\nu(\alpha)\right\} = \alpha$, and where T is a central t random variable with ν degrees of freedom.

The power of the test for a given alternative $\underline{\mu(x)}$ can be written

$$P_{\underline{\mu(x)}}\left\{\frac{\sum_{i=1}^{k} c_i\bar{Y}_{i.}}{s\sqrt{\sum_{i=1}^{k} c_i^2/n}} > t_\nu(1 - \alpha)\right\} = P_{H_o}\left\{T > t_\nu(1 - \alpha) - \frac{\sum_{i=1}^{k} c_i\mu(x_i)}{s\sqrt{\sum_{i=1}^{k} c_i^2/n}}\right.$$

where

$$T \equiv \frac{\sum_{i=1}^{k} c_i\left(\bar{Y}_{i.} - \mu(x_i)\right)}{s\sqrt{\sum_{i=1}^{k} c_i^2/n}} .$$

Rewriting power in terms of a convenient measure of dispersion and the correlation between the contrast $\underset{\sim}{c}$ and the alternative $\underline{\mu(x)}$ yields:

$$\text{Power} = P\left\{T > t_\nu(1-\alpha) - r_{\underset{\sim}{c}\underline{\mu(x)}} \Delta \frac{\sigma}{s} \sqrt{n}\right\}$$

where

$$r_{\underset{\sim}{c}\underline{\mu(x)}} \equiv \frac{\sum_{i=1}^{k} c_i \mu(x_i)}{\sqrt{\sum_{i=1}^{k} c_i^2} \sqrt{\sum_{i=1}^{k} [\mu(x_i) - \overline{\mu(x_.)}]^2}}$$

denotes the formal correlation coefficient between $\underset{\sim}{c}$ and the alternative $\underline{\mu(x)}$ for the chosen set of $\{x_1, \cdots, x_k\}$ and

$$\Delta \equiv \frac{\sqrt{\sum_{i=1}^{k} [\mu(x_i) - \overline{\mu(x_.)}]^2}}{\sigma}$$

is the measure of dispersion of the true means. It is now obvious that the power of the test for a fixed Δ is increasing in $r_{\underset{\sim}{c}\underline{\mu(x)}}$ and $\sqrt{n}$. Hence the maximin test is constructed by choosing $\underset{\sim}{c}$ such that the minimum $r_{\underset{\sim}{c}\underline{\mu(x)}}$ for $\underline{\mu(x)} \in H_a$ is maximized. The maximin contrast $\underset{\sim}{c}$ must satisfy

$$\max_{\underset{\sim}{c}^* \in H_a} \min_{\underline{\mu(x)} \in H_a} r_{\underset{\sim}{c}^*\underline{\mu(x)}} = \min_{\underline{\mu(x)} \in H_a} r_{\underset{\sim}{c}\underline{\mu(x)}} \quad . \tag{2}$$

The maximin contrast $\underset{\sim}{c}$, satisfying (2), is a function of the design points $\{x_1, x_2, \cdots, x_k\}$ and is given by the following equation:

$$\left.\begin{aligned}
c_1 &= -\frac{D_1 - D_2}{x_1 - x_2} \\
c_i &= \frac{D_{i-1} - D_i}{x_{i-1} - x_i} - \frac{D_i - D_{i+1}}{x_i - x_{i+1}}, \quad i = 2, \cdots, k-1 \\
c_k &= \frac{D_{k-1} - D_k}{x_{k-1} - x_k}
\end{aligned}\right\} \tag{3}$$

where

$$D_i = \sqrt{\sum_{j=1}^{k} a_{ij}^2 - \frac{1}{k}\left(\sum_{j=1}^{k} a_{ij}\right)^2} \quad i = 1, \cdots, k$$

and

$$A = (a_{ij}) = \begin{bmatrix}
x_1 & x_1 & x_1 & \cdots & x_1 & x_1 \\
x_1 & x_2 & x_2 & \cdots & x_2 & x_2 \\
x_1 & x_2 & x_3 & \cdots & x_3 & x_3 \\
 & \vdots & & & \vdots & \\
x_1 & x_2 & x_3 & \cdots & x_{k-1} & x_{k-1} \\
x_1 & x_2 & x_3 & \cdots & x_{k-1} & x_k
\end{bmatrix}_{k \times k} .$$

A computer program to calculate these coefficients for any design $\{x_1, \cdots, x_k\}$ is given in appendix A.

IV. APPLICATION OF THE TEST IN THE FOUR CASES OF MONOTONICITY OF MEANS AND SLOPE

The contrast $\underline{c}$ in equation (3) is for testing for alternatives with monotone increasing **trend** and monotone decreasing slope ($\mu(x)\uparrow$, $\mu(x)'\downarrow$) seen in Figure 2.a. and defined analytically by

$$H_a: \frac{\mu(x_2) - \mu(x_1)}{x_2 - x_1} \geq \cdots \geq \frac{\mu(x_k) - \mu(x_{k-1})}{x_k - x_{k-1}} \geq 0.$$

The other cases of alternatives shown in Figure 2 can be written as simple functions of $\underline{c}$.

For monotone decreasing curve with monotone decreasing slope, $(\mu(x) \downarrow,\ \mu(x)' \downarrow)$, shown in Figure 2.b. and defined analytically by

$$H_b:\ 0 \geq \frac{\mu(x_2) - \mu(x_1)}{x_2 - x_1} \geq \cdots \geq \frac{\mu(x_k) - \mu(x_{k-1})}{x_k - x_{k-1}},$$

the test statistic is based on the contrast $\underline{c}_b$, where $c_{bi} \equiv c_{k-i+1}$, $i = 1, \cdots, k$.

For monotone decreasing curve and monotone increasing slope, $(\mu(x) \downarrow,\ \mu(x)' \uparrow)$, shown in Figure 2.c. and defined analytically by

$$H_c: \frac{\mu(x_2) - \mu(x_1)}{x_2 - x_1} \leq \cdots \leq \frac{\mu(x_k) - \mu(x_{k-1})}{x_k - x_{k-1}} \leq 0,$$

the test statistic is based on the contrast $\underline{c}_c$, where $c_{ci} \equiv -c_i$, $i = 1, \cdots, k$.

For monotone increasing curve and monotone increasing slope, $(\mu(x) \uparrow,\ \mu(x)' \uparrow)$, shown in Figure 2.d. and defined analytically by

$$H_d:\ 0 \leq \frac{\mu(x_2) - \mu(x_1)}{x_2 - x_1} \leq \cdots \leq \frac{\mu(x_k) - \mu(x_{k-1})}{x_k - x_{k-1}},$$

the test statistic is based on the contrast $\underline{c}_d$, where $c_{di} \equiv -c_{k-i+1}$, $i = 1, \cdots, k$.

For each of the contrasts defined above, the α-level test of H_o is defined by Equation (1).

V. DERIVATION OF THE TEST

The alternative region H_a, is defined by the k - 1 inequalities,

$$\frac{\mu(x_2) - \mu(x_1)}{x_2 - x_1} \geq \frac{\mu(x_3) - \mu(x_2)}{x_3 - x_2} \geq \cdots \geq \frac{\mu(x_k) - \mu(x_{k-1})}{x_k - x_{k-1}} \geq 0$$

(with at least one strict inequality), which can be written as $\sum_{j=1}^{k} b_{ij}\mu(x_j) \geq 0$ where $i = 2, \cdots, k$, and

$$B = (b_{ij}) = \begin{bmatrix} (x_2-x_3) & (x_3-x_1) & (x_1-x_2) & 0 & 0 & \cdots & 0 \\ 0 & (x_3-x_4) & (x_4-x_2) & (x_2-x_3) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & (x_{k-1}-x_k) & (x_k-x_{k-2}) & (x_{k-2}-x_{k-1}) \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}_{(k-1)\times k}$$

The subset H_a of the k dimensional linear subspace, R^k, is a polyhedral angle with the H_o: $\mu(x_1) = \cdots = \mu(x_k)$ equalities determining one edge, say $\underset{\sim}{e}_1$, which is a line or one-dimensional subspace at the edge of the region H_a. [Schaafsma 1966, p. 18.] The remaining k - 1 edges of the polyhedral angle, say $\underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_k$, can then be defined by the following equations:

$$\text{and} \qquad \underset{\sim}{e}_m: \begin{cases} \sum_{j=1}^{k} b_{ij} \cdot \mu(x_j) = 0, & i = 2, \cdots, k, \text{ for } i \neq m \\ \\ \sum_{j=1}^{k} b_{mj} \cdot \mu(x_j) > 0, & m = 2, \cdots, k, \end{cases} \tag{4}$$

[Schaafsma, Smid 1966, p. 1166]. Equation (4) states that letting all but one inequality be an equality, determines an extreme edge or corner of the region H_a defined by the inequalities. Since condition (4) contains only k - 2 equalities in k unknowns, two arbitrary choices can be made, so long as they satisfy the remaining inequality of (4). In what follows the arbitrary choices, $e_{m1} \equiv x_1$ and $e_{m2} \equiv x_2$ are made for simplicity of solution for $m = 2, \cdots, k$. Now solving equations (4) we obtain:

$$\underset{\sim}{e}_2 = (x_1, x_2, x_2, x_2, \cdots, x_2)$$

$$\underset{\sim}{e}_3 = (x_1, x_2, x_3, x_3, \cdots, x_3, x_3)$$

$$\underset{\sim}{e}_4 = (x_1, x_2, x_3, x_4, x_4, \cdots, x_4, x_4)$$

$$\vdots$$

$$\underset{\sim}{e}_k = (x_1, x_2, x_3, \cdots, x_{k-1}, x_k).$$

The edge $\underset{\sim}{e}_1$, determined by the k - 1 equalities of H_o and the arbitrary choice $e_{11} = x_1$ can be written $\underset{\sim}{e}_1 = (x_1, x_1, \cdots, x_1)$.

The alternative region H_a in R^k is a convex set with all elements determined by positive multiples of $\underset{\sim}{e}_1, \cdots, \underset{\sim}{e}_k$. The maximin-r test is thus determined by finding the $\underset{\sim}{c}$ vector, interior to this region, which maximizes the minimum $r_{\underset{\sim}{c}\,\underline{\mu(x)}}$. Thus $\underset{\sim}{c}$ is the contrast or vector which makes equal angles with each edge of H_a [Abelson, Tukey 1963].

For any k, $\underset{\sim}{c}$ can be found by solving the k - 2 equations $r_{\underset{\sim}{c}\underset{\sim}{e}_2} = r_{\underset{\sim}{c}\underset{\sim}{e}_3} = \cdots = r_{\underset{\sim}{c}\underset{\sim}{e}_k}$ and $\sum_{j=1}^{k} c_j = 0$. The normalization of $\underset{\sim}{c}$, $\sum_{j=1}^{k} c_j^2$, can be chosen arbitrarily. An easy method of solving for $\underset{\sim}{c}$ is to let the normalization be arbitrary and solve the k - 1 equations:

$$\left.\begin{aligned} &\sum_{j=1}^{k} c_j e_{ij} = \sqrt{\sum_{j=1}^{k} e_{ij}^2 - \frac{1}{k}\Bigl(\sum_{j=1}^{k} e_{ij}\Bigr)^2} \quad i = 2, \cdots, k \\ \text{and} \\ &\sum_{j=1}^{k} c_j = 0, \end{aligned}\right\} \tag{5}$$

the contrast equation. The solution of (5) yields the test contrast given in equation (3) as shown below. The minimum value of r for any alternative in H_a is therefore

$$\frac{1}{\sqrt{\sum_{j=1}^{k} c_j^2}}$$

by equation (5) and the definition of r.

Let

$$D_i \equiv \sqrt{\sum_{j=1}^{k} e_{ij}^2 - \frac{1}{k}\Bigl(\sum_{j=1}^{k} e_{ij}\Bigr)^2} \quad \text{for } i = 1, \cdots, k$$

be the square root of the corrected sum of squares of the corner vectors. Note that $D_1 = 0$ but is included for completeness. Now from equation (5), the following must be solved for $\underline{e}_2$:

$$\sum_{j=1}^{k} c_j e_{2j} = D_2$$

$$\Rightarrow c_1 x_1 + (c_2 + c_3 + \cdots + c_k) x_2 = D_2 \ . \tag{6}$$

For $\underset{\sim}{e}_3$, equation (5) implies:

$$\sum_{j=1}^{k} c_j e_{3j} = D_3$$

$$\Rightarrow c_1x_1 + c_2x_2 + (c_3 + c_4 + \cdots + c_k)x_3 = D_3 \ . \tag{7}$$

For $\underset{\sim}{e}_4$:

$$c_1x_1 + c_2x_2 + c_3x_3 + (c_4 + c_5 + \cdots + c_k)x_4 = D_4 \ . \tag{8}$$

Now since $\sum_{j=1}^{k} c_j = 0$, thus $c_2 + \cdots + c_k = -c_1$ and $c_3 + c_4 + \cdots + c_k = -c_1 - c_2$, etc. and therefore equations (6) - (8) can be written as:

$$c_1x_1 + (-c_1)x_2 = D_2 \tag{9}$$

$$c_1x_1 + c_2x_2 + (-c_1 - c_2)x_3 = D_3 \tag{10}$$

$$c_1x_1 + c_2x_2 + c_3x_3 + (-c_1 - c_2 - c_3)x_4 = D_4 \ . \tag{11}$$

Now (9) implies that $c_1 = \dfrac{D_2}{x_1 - x_2}$. Solving (10) for c_2 yields:

$$c_2(x_2 - x_3) + c_1(x_1 - x_3) = D_3$$

$$\Rightarrow \quad c_2(x_2 - x_3) + c_1(x_1 - x_2) + c_1(x_2 - x_3) = D_3$$

$$\Rightarrow \quad c_2(x_2 - x_3) + D_2 + \frac{D_2(x_2 - x_3)}{(x_1 - x_2)} = D_3$$

$$\Rightarrow \quad c_2 = - \frac{D_2}{x_1 - x_2} - \frac{D_2 - D_3}{x_2 - x_3} \ .$$

Solving (11) for c_3 yields:

$$c_1(x_1 - x_4) + c_2(x_2 - x_4) + c_3(x_3 - x_4) = D_4$$

$$\Rightarrow \quad \frac{D_2}{x_1 - x_2}(x_1 - x_4) + \left[\frac{-D_2}{x_1 - x_2} - \frac{D_2 - D_3}{x_2 - x_3}\right](x_2 - x_4) + c_3(x_3 - x_4) = D_4$$

$$\Rightarrow \quad \frac{-D_2(x_3 - x_4)}{x_2 - x_3} + D_3 + \frac{D_3(x_3 - x_4)}{x_2 - x_3} + c_3(x_3 - x_4) = D_4$$

$$\Rightarrow \quad c_3 = \frac{D_2 - D_3}{x_2 - x_3} - \frac{D_3 - D_4}{x_3 - x_4} \quad .$$

Similarly,

$$c_4 = \frac{D_3 - D_4}{x_3 - x_4} - \frac{D_4 - D_5}{x_4 - x_5} \quad , \text{ etc.}$$

Now, assuming

$$c_{i-1} = \frac{D_{i-2} - D_{i-1}}{x_{i-2} - x_{i-1}} - \frac{D_{i-1} - D_i}{x_{i-1} - x_i} \; ,$$

it is sufficient to prove that

$$c_i = \frac{D_{i-1} - D_i}{x_{i-1} - x_i} - \frac{D_i - D_{i+1}}{x_i - x_{i+1}} \; .$$

From (5)

$$c_1x_1 + c_2x_2 + \cdots + c_ix_i + (c_{i+1} + \cdots + c_k)x_{i+1} = D_{i+1}$$

$$\Rightarrow \quad c_1x_1 + c_2x_2 + \cdots + c_ix_i + (-c_1 - c_2 - \cdots - c_i)x_{i+1} = D_{i+1}$$

$$\Rightarrow \quad c_1(x_1 - x_{i+1}) + c_2(x_2 - x_{i+1}) + \cdots + c_i(x_i - x_{i+1}) = D_{i+1} \; . \qquad (12)$$

Now since

$$c_1 = \frac{D_2}{x_1 - x_2}$$

$$c_2 = \frac{-D_2}{x_1 - x_2} - \frac{D_2 - D_3}{x_2 - x_3}$$

$$c_3 = \frac{D_2 - D_3}{x_2 - x_3} - \frac{D_3 - D_4}{x_3 - x_4}$$

$$\vdots$$

$$c_{i-1} = \frac{D_{i-2} - D_{i-1}}{x_{i-2} - x_{i-1}} - \frac{D_{i-1} - D_i}{x_{i-1} - x_i} ,$$

Hence (12) can be written:

$$\begin{aligned}
& \frac{D_2}{x_1 - x_2} \cdot (x_1 - x_2 + x_2 - x_{i+1}) \\
+ & \frac{-D_2}{x_1 - x_2} \cdot (x_2 - x_{i+1}) - \frac{D_2 - D_3}{x_2 - x_3} \cdot (x_2 - x_3 + x_3 - x_{i+1}) \\
+ & \frac{D_2 - D_3}{x_2 - x_3} \cdot (x_3 - x_{i+1}) - \frac{D_3 - D_4}{x_3 - x_4} \cdot (x_3 - x_4 + x_4 - x_{i+1}) \\
& \qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
+ & \frac{D_{i-2} - D_{i-1}}{x_{i-2} - x_{i-1}} (x_{i-1} - x_{i+1}) - \frac{D_{i-1} - D_i}{x_{i-1} - x_i} (x_{i-1} - x_i + x_i - x_{i+1}) \\
& + c_i(x_i - x_{i+1}) = D_{i+1}
\end{aligned}$$

$$\Rightarrow \quad D_2 - (D_2 - D_3) - (D_3 - D_4) - \cdots - (D_{i-1} - D_i) - \frac{D_{i-1} - D_i}{x_{i-1} - x_i}(x_i - x_{i+1})$$

$$+ c_i(x_i - x_{i+1}) = D_{i+1}$$

$$\Rightarrow \quad D_i - \frac{D_{i-1} - D_i}{x_{i-1} - x_i}(x_i - x_{i+1}) + c_i(x_i - x_{i+1}) = D_{i+1}$$

$$\Rightarrow \quad c_i = \frac{D_{i-1} - D_i}{x_{i-1} - x_i} - \frac{D_i - D_{i+1}}{x_i - x_{i+1}} \quad \text{for all } i < k.$$

Q.E.D.

Now since $c_k = -c_1 - c_2 - \cdots - c_{k-1}$,

$$\Rightarrow \quad c_k = \frac{D_{k-1} - D_k}{x_{k-1} - x_k}.$$

By definition, $D_1 = 0$, hence we can write:

$$c_1 = \frac{D_2}{x_1 - x_2} = -\frac{D_1 - D_2}{x_1 - x_2}$$

and

$$c_2 = \frac{-D_2}{x_1 - x_2} - \frac{D_2 - D_3}{x_2 - x_3} = \frac{D_1 - D_2}{x_1 - x_2} - \frac{D_2 - D_3}{x_2 - x_3}.$$

Thus the form of equation (3) is proved.

VI. LITERATURE REVIEW

Tests of equality of means, H_o: $\mu_1 = \mu_2 = \cdots = \mu_k$ against monotone alternatives, H_m: $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_k$ (with at least one inequality strict) have been studied extensively. Abelson and Tukey [1963] derived a test based on a contrast of the means for the case where the means are normally distributed and have equal variances. They also described procedures for finding the test for partial ordering of the μ_i's.

Schaafsma and Smid [1966] generalized the method of Abelson and Tukey by showing that for testing any hypothesis defined by r linear combinations of the means,

$$H_o\colon b_{oh} + \sum_{i=1}^{k} b_{ih} \cdot \mu_i = 0, \; h = 1, \cdots, r,$$

against alternatives defined by the corresponding r inequalities,

$$H_a\colon b_{oh} + \sum_{i=1}^{k} b_{ih} \cdot \mu_i \geq 0, \; h = 1, \cdots, r,$$

there exists a most stringent among somewhere most powerful (MSSMP) test. Theorems are given which prove the existence of MSSMP tests for cases where σ^2 is known and MSSMP similar tests where σ^2 is unknown. Also, for two-sided alternatives, MSSMP unbiased tests are described. Application of these methods is made to the problem of equality of means against an increasing trend [Schaafsma, Smid 1966]. Other applications including the combination of independent tests, testing additivity in a two-way layout, and testing goodness-of-fit, all against restricted alternatives have been described [Schaafsma 1966].

Another approach to testing $H_o\colon \mu_1 = \mu_2 = \cdots = \mu_k$ versus $H_m\colon \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k$ utilizes the likelihood ratio principle [Bartholomew 1959a,b, 1961a,b]. The test statistics, $\bar{\chi}^2$ or $\bar{E}^2$, for σ^2 known or unknown, respectively, are derived as likelihood ratio test statistics. This method requires finding the maximum likelihood estimators of the μ_i's under H_m, which are the solution of a quadratic programming problem. The distribution of $\bar{X}^2$ under the null hypothesis for σ^2 known is a known mixture of chi-square distributions. Similarly for σ^2 unknown, the null distribution of $\bar{E}^2$ is a known mixture of Beta distributions.

Kudô [1963] gave a general multivariate formulation of the one-sided test assuming the covariance matrix is known. Tests of H_o versus H_m described above can be transformed such that this formulation is a generalization of Bartholomew's work and yields the same solutions in specific cases.

Shorack [1967] extended Bartholomew's work for two-way layouts and higher order complete and incomplete designs.

A complete summary of these tests has been published [Barlow - Brunk 1972, Chapters 3, 4]. Power comparisons between tests based on contrasts and likelihood ratio tests are made and selection among the tests is discussed. In addition, distribution-free tests are reviewed and discussed.

Although none of the above mentioned literature utilizes the quantitative information contained in the spacing of the x_i's, Barlow - Brunk [1972, p. 190] state: "If additional prior information about the spacing as well as the ordering is available a T-statistic, with scores chosen according to the spacing, should give a power near to the values tabulated for T(max). In these circumstances T may be preferred to $\bar{X}^2$." The results of this study support the above conjecture that the minimum power over all possible alternatives in H_a, defined using the spacing of the x_i's, is closer to the maximum power attainable (denoted T(max) above), than is achieved by the test for alternatives defined only by the rankings of the x_i's.

VII. POWER COMPARISONS WITH TESTS FOR MONOTONE ALTERNATIVES

Comparison is made of the test derived in this report for monotone concave alternatives, H_{mc}, against the existing tests for monotone alternatives, H_m, derived by Abelson and Tukey [1963] and Bartholomew [1959, 1961].

Since our test depends on the spacing of the x_i's, power calculations will be made for equally spaced x_i's, $\underline{x} = (1, 2, 3, 4, \cdots, k)$, and for x_i's in a geometric series which are equally spaced on the log scale, $\underline{x} = (1, 2, 4, 8, \cdots, 2^{k-1})$ which is commonly used in dose-response studies. These alternatives are denoted $H_{mc}(x)$ and $H_{mc}(\log x)$, respectively. The power will be calculated for $\Delta = 0(1)4$, $n = 1$, $\alpha = .05$, and σ assumed known. Thus from section III, the power can be written $\Phi(r \cdot \Delta - z_{1-\alpha})$, where Δ is the measure of dispersion and $\Phi(u)$ is the standard normal integral. The power range is determined by $r = 1$ and $r =$ maximin r for the most and least favorable configurations, respectively, of the means. Table 1 gives the power range for $k = 3(1)6(2)12$.

Power comparisons for $\Delta = 0(1)4$ and for $k = 3, 4, 12$ between the $\bar{\chi}^2$ test of Bartholomew [1959a,b] and the maximin r test against H_m of Abelson and Tukey [1963] were taken from Barlow - Brunk [Table 4.1, p. 189].

The minimum power of the maximin t-test for alternatives in H_{mc} is higher for all k and Δ than the minimum power for the maximin t-test against all monotone alternatives H_m, regardless of the spacing of the treatment levels. See Table 1. This is expected since the alternative space H_{mc} is a proper subset of H_m and hence

$$\min_{\underline{\mu(x)} \in H_{mc}} r_{\underline{c} \cdot \underline{\mu(x)}} \geq \min_{\underline{\mu} \in H_m} r_{\underline{c} \cdot \underline{\mu}} .$$

A view of the alternatives H_m and H_{mc} for k = 3 is given in Figure 3, which illustrates the relative sizes of these regions.

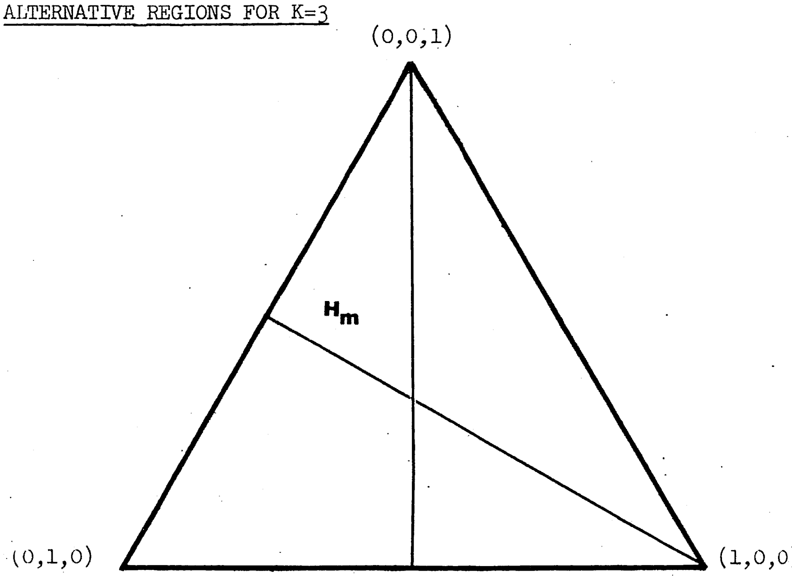
Table 1: Power Comparison of t-test Developed Herein for Monotone Concave Alternatives H_{mc}, t-test for Monotone Alternatives H_m, and $\bar{\chi}^2$ Test for Monotone Alternatives

k	Test	Power	Δ = 0	Δ = 1	Δ = 2	Δ = 3	Δ = 4
3	t-test	max	.050	.260	.639	.912	.991
		min ϵ $H_{mc}(x)$	.050	.249	.613	.895	.987
		min ϵ $H_{mc}(\log x)$	.050	.240	.591	.878	.982
		min ϵ H_m	.050	.218	.535	.830	.966
	$\bar{\chi}^2$	max ϵ H_m	.050	.244	.605	.892	.987
		min ϵ H_m	.050	.221	.569	.872	.983
4	t-test	max	.050	.260	.639	.912	.991
		min ϵ $H_{mc}(x)$	.050	.240	.592	.879	.983
		min ϵ $H_{mc}(\log x)$	.050	.226	.555	.848	.973
		min ϵ H_m	.050	.201	.488	.781	.943
	$\bar{\chi}^2$	max ϵ H_m	.050	.238	.594	.885	.985
		min ϵ H_m	.050	.202	.531	.849	.977
5	t-test	max	.050	.260	.639	.912	.991
		min ϵ $H_{mc}(x)$	.050	.234	.577	.867	.979
		min ϵ $H_{mc}(\log x)$	.050	.216	.529	.824	.963
		min ϵ H_m	.050	.191	.460	.749	.926
6	t-test	max	.050	.260	.639	.912	.991
		min ϵ $H_{mc}(x)$	.050	.230	.566	.858	.976
		min ϵ $H_{mc}(\log x)$	.050	.209	.509	.804	.955
		min ϵ H_m	.050	.184	.440	.724	.910
8	t-test	max	.050	.260	.639	.912	.991
		min ϵ $H_{mc}(x)$	.050	.223	.549	.843	.971
		min ϵ $H_{mc}(\log x)$	.050	.198	.480	.772	.939
		min ϵ H_m	.050	.176	.414	.691	.887
10	t-test	max	.050	.260	.639	.912	.991
		min ϵ $H_{mc}(x)$	.050	.219	.537	.832	.967
		min ϵ $H_{mc}(\log x)$	.050	.191	.458	.747	.924
		min ϵ H_m	.050	.170	.397	.666	.869
12	t-test	max	.050	.260	.639	.912	.991
		min ϵ $H_{mc}(x)$	.050	.216	.528	.823	.963
		min ϵ $H_{mc}(\log x)$	.050	.185	.442	.727	.912
		min ϵ H_m	.050	.166	.385	.649	.855
	$\bar{\chi}^2$	min ϵ H_m (approx.)	(.067)	(.159)	(.414)	(.766)	(.963)

NOTE: k = number of means $\quad \Delta^2 = \sum_{i=1}^{k} \left(\mu(x_i) - \overline{\mu(x_.)}\right)^2/\sigma^2$

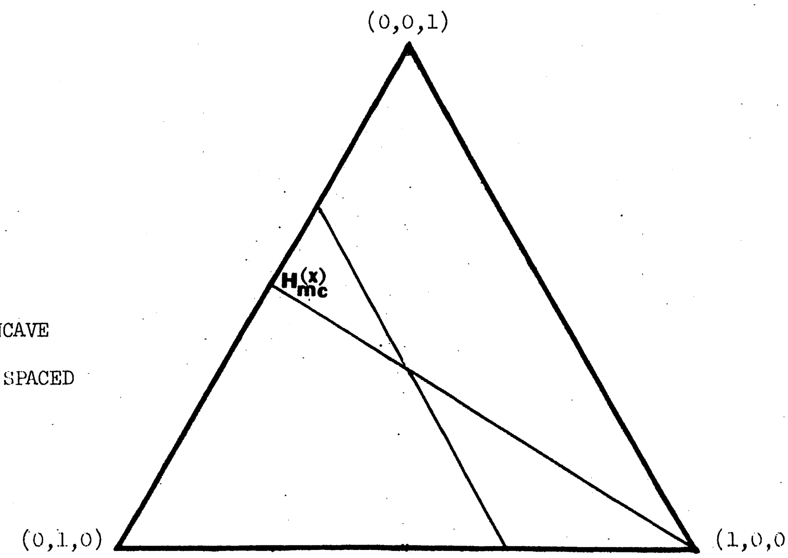
FIGURE 3 ALTERNATIVE REGIONS FOR K=3

MONOTONE



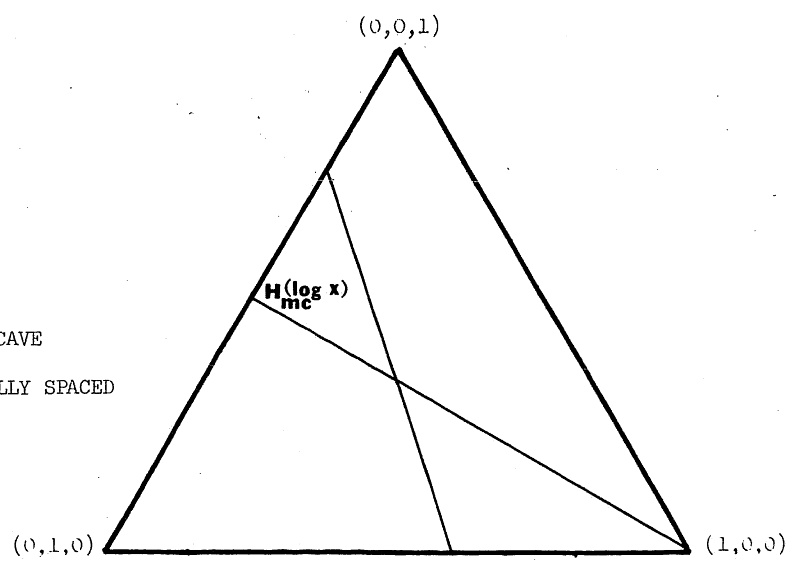
MONOTONE CONCAVE

X's EQUALLY SPACED



MONOTONE CONCAVE

LOG X's EQUALLY SPACED



Comparison of the maximin t-test with the $\bar{\chi}^2$ test is less straightforward and exact values are available only for $k = 3$ and 4. The maximin t-test for monotone alternatives H_m has minimum power less than the $\bar{\chi}^2$ test but maximum power higher than the $\bar{\chi}^2$ test for monotone alternatives. Thus, there is no clear choice between these two tests on the basis of power.

When the additional assumption of monotone slope is justified, the maximin t-test for alternatives $H_{mc}(x)$ has minimum power higher than the minimum power of the $\bar{\chi}^2$ test for H_m alternatives for all Δ values. For $\Delta = 1$ the minimum power of the maximin test for $H_{mc}(x)$ is even higher than the maximum power of the $\bar{\chi}^2$ test for H_m. For alternatives $H_{mc}(\log x)$, the minimum power of the maximin t-test is higher than the minimum power of the $\bar{\chi}^2$ test for all cases considered except ($k = 4$, $\Delta = 4$) and perhaps ($k = 12$, $\Delta \geq 3$).

Although the minimum power of the maximin t-test for H_{mc} alternatives depends on the spacing of the x_i's, the maximum power is always higher than the power of the $\bar{\chi}^2$ test for H_m alternatives and the minimum power is in or near the range of power for the $\bar{\chi}^2$ test. Therefore if it is known that the alternatives are monotone with monotone slope, the maximin t-test described in this report is preferable on the basis of power to the existing tests for monotone alternatives.

VIII. APPLICATION OF THE TEST FOR MONOTONE CONCAVE ALTERNATIVES; AN EXAMPLE

Data is given in Table 2 from a study of animal weight gain where treatments are levels of antibiotics in the feed. The treatment levels, x_i's, and responses, Y_{ij}'s, are given in arbitrary units. The usual analysis of variance for a randomized complete block design indicates that the treatment effects are not significant at the .05 level.

The data is also analyzed by the maximin test for monotone concave alternatives given in equation (1) using coefficients calculated by the algorithm given in appendix (D) for $\underline{x}$ = (0, 1, 2, 4, 8, 16). If the test rejects at the .05 level, we remove the first treatment and retest the remaining data until the test does not reject. This procedure is utilized in the search for the treatment level beyond which the data indicates a plateau. Table 3 gives the results of this sequence of tests.

Table 2: An Example of a Dose-Response Experiment in a Randomized Complete Block Design

Treatment Dose Level (x_i)	Response (Y_{ij}) Replicates 1	2	3	4	Treatment Means ($\bar{Y}_{i.}$)
0	8.07	7.58	8.29	8.19	8.033
1	8.28	8.27	8.32	8.61	8.370
2	8.96	8.50	8.53	8.63	8.655
4	8.00	9.44	9.01	8.75	8.800
8	9.18	7.65	9.75	8.04	8.655
16	9.13	8.86	9.73	9.02	9.185

ANOVA

	d.f.	SS	MS	F
Replicates	3	.984	.328	
Treatments	5	3.047	.609	2.332
Residual	15	3.917	.261	

$$F_{5,15}(.90) = 2.27$$

$$F_{5,15}(.95) = 2.90$$

Table 3: T-test Values of Maximin Test for Data in Table 2. Coefficients for Equation (1) are shown in Appendix 1d., s = .511 with 15 degrees of freedom, and n = 4.

Treatment levels	Calculated t (Equ. (1))
0, 1, 2, 4, 8, 16	3.178
1, 2, 4, 8, 16	2.122
2, 4, 8, 16	1.281

Level .05 critical point: $t_{15}(.95) = 1.753$

For $\underline{x} = (0, 1, 2, 4, 8, 16)$ and $\underline{x} = (1, 2, 4, 8, 16)$, the test rejects at the .05 level indicating that in the treatment range (0 - 16) or (1 - 16), the means are not homogeneous. However, the test of $H_o : \mu(2) = \mu(4) = \mu(8) = \mu(16)$ does not reject at .05 indicating that these means are not monotone concave and hence are consistent with the hypothesis that they are homogeneous.

IX. ADDENDUM: POWER COMPARISON OF A LIKELIHOOD RATIO TEST FOR MONOTONE CONCAVE ALTERNATIVES ($\bar{\chi}^2_{mc}$).

A likelihood ratio approach to the problem of testing H_o versus H_{mc} has also been pursued. The test statistic, $\bar{\chi}^2_{mc}$, is the corrected sum of squares among the maximum likelihood estimates under H_{mc}. The null distribution of $\bar{\chi}^2_{mc}$ is a mixture of central chi-square distributions, with mixing probabilities which depend only on the specified treatment design point, $x = (x_1, \cdots, x_k)$. In particular, for k = 3,

$$P_{H_o}\left\{ \bar{\chi}^2_{mc} \leq \gamma \right\} = \sum_{i=0}^{2} p_i \cdot P\left\{ \chi^2_i \leq \gamma \right\}$$

where

$$p_0 = \frac{7}{12} - \frac{1}{2\pi} \tan^{-1} \left\{ \frac{- x_1 - x_2 + 2x_3}{\sqrt{3}\ (- x_1 + x_2)} \right\}$$

$$p_1 = \frac{5}{12} + \frac{1}{2\pi} \tan^{-1} \left\{ \frac{- x_1 - x_2 + 2x_3}{\sqrt{3}\ (- x_1 + x_2)} \right\} - \frac{1}{2\pi} \tan^{-1} \left\{ \frac{\sqrt{3}\ (- x_2 + x_3)}{- 2x_1 + x_2 + x_3} \right\}$$

$$p_2 = \frac{1}{2\pi} \tan^{-1} \left\{ \frac{\sqrt{3}\ (- x_2 + x_3)}{- 2x_1 + x_2 + x_3} \right\}$$

and where $P\{\chi_0^2 = 0\} = 1$ and for $\nu > 0$, χ_ν^2 is a central chi-square random variable with ν degrees of freedom.

Table 4 compares the power of the $\bar{\chi}^2_{mc}$ test to the maximin t-test for monotone concave alternatives for the case $k = 3$. Both tests depend on the treatment levels $\{x_1, x_2, x_3\}$ and power is given for equally spaced x_i's, $H_{mc}(x)$ and for x_i's in a geometric series $\{1, 2, 4\}$ which are equally spaced on the log scale and denoted $H_{mc}(\log x)$. The results show the power range of the $\bar{\chi}^2_{mc}$ test being

Table 4: Power Comparison of the Maximin t-test for Monotone Concave Alternatives to the $\bar{\chi}^2_{mc}$ Test for Monotone Concave Alternatives, For the Case $k = 3$

Alternative	Test	Power	$\Delta = 0$	$\Delta = 1$	$\Delta = 2$	$\Delta = 3$	$\Delta = 4$
$H_{mc}(x)$	t-test	max	.050	.260	.639	.912	.991
	$\bar{\chi}^2_{mc}$	max	.050	.255	.629	.906	.989
	$\bar{\chi}^2_{mc}$	min	.050	.247	.613	.898	.988
	t-test	min	.050	.249	.613	.895	.987
$H_{mc}(\log x)$	t-test	max	.050	.260	.639	.912	.991
	$\bar{\chi}^2_{mc}$	max	.050	.252	.622	.902	.988
	$\bar{\chi}^2_{mc}$	min	.050	.238	.598	.889	.986
	t-test	min	.050	.226	.555	.848	.973

NOTE: $H_{mc}(x)$ and $H_{mc}(\log x)$ denote monotone concave alternatives for equally spaced x_i's and equally spaced x_i's on the log scale.

contained within the power range of the maximin t-test except for the case $\Delta = 1$ when the x_i's are evenly spaced.

The $\bar{\chi}^2_{mc}$ test is compared to the $\bar{\chi}^2$ test for monotone alternatives in Table 5. As expected, the power range is higher for the $\bar{\chi}^2_{mc}$ test and for evenly spaced x_i's the minimum power of the $\bar{\chi}^2_{mc}$ test is higher than the maximum power of the $\bar{\chi}^2$ test.

Table 5: Power Comparison of the $\bar{\chi}^2$ Test for Monotone Alternatives to the $\bar{\chi}^2_{mc}$ Test for Monotone Concave Alternatives, for the Case $k = 3$

Test	Power	$\Delta = 0$	$\Delta = 1$	$\Delta = 2$	$\Delta = 3$	$\Delta = 4$
$\bar{\chi}^2_{mc}$	max ϵ $H_{mc}(x)$	.050	.255	.629	.906	.989
	min ϵ $H_{mc}(x)$	.050	.247	.613	.898	.988
	max ϵ $H_{mc}(\log x)$	.050	.252	.622	.902	.988
	min ϵ $H_{mc}(\log x)$	.050	.238	.598	.889	.986
$\bar{\chi}^2$	max ϵ H_m	.050	.244	.605	.892	.987
	min ϵ H_m	.050	.221	.569	.872	.983

NOTE: $H_{mc}(x)$ and $H_{mc}(\log x)$ denote monotone concave alternatives for equally spaced x_i's and equally spaced x_i's on the log scale.

These results further support the suggestion that for quantitative treatments where the response function is known to be monotone with monotone slope, the test procedure based on this prior information is more powerful than previously available tests.

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APPENDIX A

```
C     PROGRAM TO CALCULATE MAXIMIN CONTRASTS
      DIMENSION X(20), C(20)
C     INPUT SETS OF CARDS
C     I2 - NO. OF X'S
C     8F10.0 - X'S
    1 READ(5,99) NO
   99 FORMAT(I2)
      IF(NO.EQ.0) CALL EXIT
      READ(5,100)(X(I),I=1,NO)
  100 FORMAT (8F10.0)
      WRITE(6,104) (X(I),I=1,NO)
  104 FORMAT('1SPACING OF X''S IN TREATMENT DESIGN'/2(1X,10F8.2/))
C     COMPUTE CONTRAST FOR ALL X'S AND THEN ITERATIVELY DROP THE FIRST X
      NO2 = NO - 2
      DO 50 K=1,NO2
      NUM = NO - K + 1
      CALL MC(NUM,X(K),C(K))
      RR=0
      DO 35 J=K,NO
   35 RR=RR+C(J)*C(J)
      RR=1/RR
      R=SQRT(RR)
      WRITE(6,102) NUM,RR,R
  102 FORMAT('0NO. OF MEANS =',I4,6X 'MAXIMIN R-SQUARED = ',F9.5,6X,
     C' MAXIMIN R =',F9.5)
      WRITE(6,101) (X(I),I=K,NO)
  101 FORMAT(' X''S USED TO CALCULATE MAXIMIN CONTRASTS'/1(1X,10F8.2 ))
      WRITE(6,103) (C(J),J=K,NO)
  103 FORMAT(' VALUES OF MAXIMIN CONTRAST' /  1( ' ',10F8.4 ))
   50 CONTINUE
      GO TO 1
      END

      SUBROUTINE MC(K,X,C)
      DIMENSION X(K), C(K), A(20,20), D(20)
      K1=K-1
      DO 20 I=1,K
      SS=0
      SUM=0
      DO 10 J=1,K
      L=MINO(J,I)
C     A(I, ) ARE THE K-1 CORNER VECTORS
      A(I,J)=X(L)
      SS=SS+A(I,J)*A(I,J)
   10 SUM=SUM+A(I,J)
   20 D(I) = SQRT(SS-(SUM*SUM)/K)
C     COMPUTE CONTRAST
      C(1)=D(2)/(X(1)-X(2))
      DO 30 J=2,K1
   30 C(J)=(D(J-1)-D(J))/(X(J-1)-X(J)) - (D(J)-D(J+1))/(X(J)-X(J+1))
      C(K) = (D(K-1)-D(K))/(X(K-1)-X(K))
      RETURN
      END
```

APPENDIX B

```
SPACING OF X'S IN TREATMENT DESIGN
    1.00    2.00    3.00    4.00    5.00    6.00    7.00    8.00    9.00   10.00
   11.00   12.00

NO. OF MEANS =  12      MAXIMIN R-SQUARED =   0.73569       MAXIMIN R =  0.85772
X'S USED TO CALCULATE MAXIMIN CONTRASTS
    1.00    2.00    3.00    4.00    5.00    6.00    7.00    8.00    9.00   10.00
   11.00   12.00
VALUES OF MAXIMIN CONTRAST
 -0.9574 -0.1467 -0.1510 -0.0830 -0.0279  0.0214  0.0691  0.1186  0.1730  0.2369
  0.3177  0.4293

NO. OF MEANS =  11      MAXIMIN R-SQUARED =   0.74498       MAXIMIN R =  0.86312
X'S USED TO CALCULATE MAXIMIN CONTRASTS
    2.00    3.00    4.00    5.00    6.00    7.00    8.00    9.00   10.00   11.00
   12.00
VALUES OF MAXIMIN CONTRAST
 -0.9535 -0.1380 -0.1388 -0.0671 -0.0075  0.0474  0.1024  0.1619  0.2313  0.3192
  0.4427

NO. OF MEANS =  10      MAXIMIN R-SQUARED =   0.75549       MAXIMIN R =  0.86919
X'S USED TO CALCULATE MAXIMIN CONTRASTS
    3.00    4.00    5.00    6.00    7.00    8.00    9.00   10.00   11.00   12.00
VALUES OF MAXIMIN CONTRAST
 -0.9487 -0.1275 -0.1239 -0.0471  0.0185  0.0812  0.1473  0.2232  0.3196  0.4574

NO. OF MEANS =   9      MAXIMIN R-SQUARED =   0.76753       MAXIMIN R =  0.87609
X'S USED TO CALCULATE MAXIMIN CONTRASTS
    4.00    5.00    6.00    7.00    8.00    9.00   10.00   11.00   12.00
VALUES OF MAXIMIN CONTRAST
 -0.9428 -0.1194 -0.1051 -0.0216  0.0528  0.1276  0.2118  0.3182  0.4735

NO. OF MEANS =   8      MAXIMIN R-SQUARED =   0.78159       MAXIMIN R =  0.88408
X'S USED TO CALCULATE MAXIMIN CONTRASTS
    5.00    6.00    7.00    8.00    9.00   10.00   11.00   12.00
VALUES OF MAXIMIN CONTRAST
 -0.9354 -0.0977 -0.0806  0.0126  0.1093  0.1953  0.3143  0.4912

NO. OF MEANS =   7      MAXIMIN R-SQUARED =   0.79833       MAXIMIN R =  0.89349
X'S USED TO CALCULATE MAXIMIN CONTRASTS
    6.00    7.00    8.00    9.00   10.00   11.00   12.00
VALUES OF MAXIMIN CONTRAST
 -0.9258 -0.0756 -0.0474  0.0608  0.1712  0.3062  0.5106

NO. OF MEANS =   6      MAXIMIN R-SQUARED =   0.81883       MAXIMIN R =  0.90489
X'S USED TO CALCULATE MAXIMIN CONTRASTS
    7.00    8.00    9.00   10.00   11.00   12.00
VALUES OF MAXIMIN CONTRAST
 -0.9130 -0.0449  0.0003  0.1345  0.2913  0.5318

NO. OF MEANS =   5      MAXIMIN R-SQUARED =   0.84521       MAXIMIN R =  0.91935
X'S USED TO CALCULATE MAXIMIN CONTRASTS
    8.00    9.00   10.00   11.00   12.00
VALUES OF MAXIMIN CONTRAST
 -0.8945  0.0000  0.0757  0.2642  0.5546

NO. OF MEANS =   4      MAXIMIN R-SQUARED =   0.88085       MAXIMIN R =  0.93854
X'S USED TO CALCULATE MAXIMIN CONTRASTS
    9.00   10.00   11.00   12.00
VALUES OF MAXIMIN CONTRAST
 -0.8660  0.0737  0.2145  0.5778

NO. OF MEANS =   3      MAXIMIN R-SQUARED =   0.93296       MAXIMIN R =  0.96590
X'S USED TO CALCULATE MAXIMIN CONTRASTS
   10.00   11.00   12.00
VALUES OF MAXIMIN CONTRAST
 -0.8165  0.2189  0.5977
```

APPENDIX C

```
SPACING OF X'S IN TREATMENT DESIGN
    1.00    2.00    4.00    8.00   16.00   32.00   64.00  128.00  256.00  512.00
 1024.00 2048.00

NO. OF MEANS =  12       MAXIMIN R-SQUARED =    0.56138        MAXIMIN R =  0.74925
X'S USED TO CALCULATE MAXIMIN CONTRASTS
    1.00    2.00    4.00    8.00   16.00   32.00   64.00  128.00  256.00  512.00
 1024.00 2048.00
VALUES OF MAXIMIN CONTRAST
 -0.9574 -0.2159 -0.1952 -0.1260 -0.0705 -0.0203  0.0292  0.0820  0.1438  0.2259
  0.3607  0.7417

NO. OF MEANS =  11       MAXIMIN R-SQUARED =    0.57601        MAXIMIN R =  0.75895
X'S USED TO CALCULATE MAXIMIN CONTRASTS
    2.00    4.00    8.00   16.00   32.00   64.00  128.00  256.00  512.00 1024.00
 2048.00
VALUES OF MAXIMIN CONTRAST
 -0.9535 -0.2074 -0.1814 -0.1104 -0.0596  0.0055  0.0633  0.1290  0.2145  0.3525
  0.7385

NO. OF MEANS =  10       MAXIMIN R-SQUARED =    0.59274        MAXIMIN R =  0.76990
X'S USED TO CALCULATE MAXIMIN CONTRASTS
    4.00    8.00   16.00   32.00   64.00  128.00  256.00  512.00 1024.00 2048.00
VALUES OF MAXIMIN CONTRAST
 -0.9487 -0.1972 -0.1669 -0.0910 -0.0249  0.0397  0.1108  0.2007  0.3427  0.7348

NO. OF MEANS =   9       MAXIMIN R-SQUARED =    0.61212        MAXIMIN R =  0.78238
X'S USED TO CALCULATE MAXIMIN CONTRASTS
    8.00   16.00   32.00   64.00  128.00  256.00  512.00 1024.00 2048.00
VALUES OF MAXIMIN CONTRAST
 -0.9428 -0.1844 -0.1485 -0.0658  0.0093  0.0877  0.1834  0.3306  0.7305

NO. OF MEANS =   8       MAXIMIN R-SQUARED =    0.63489        MAXIMIN R =  0.79680
X'S USED TO CALCULATE MAXIMIN CONTRASTS
   16.00   32.00   64.00  128.00  256.00  512.00 1024.00 2048.00
VALUES OF MAXIMIN CONTRAST
 -0.9354 -0.1681 -0.1245 -0.0320  0.0576  0.1614  0.3156  0.7255

NO. OF MEANS =   7       MAXIMIN R-SQUARED =    0.66215        MAXIMIN R =  0.81373
X'S USED TO CALCULATE MAXIMIN CONTRASTS
   32.00   64.00  128.00  256.00  512.00 1024.00 2048.00
VALUES OF MAXIMIN CONTRAST
 -0.9258 -0.1466 -0.0918  0.0162  0.1324  0.2962  0.7195

NO. OF MEANS =   6       MAXIMIN R-SQUARED =    0.69558        MAXIMIN R =  0.83402
X'S USED TO CALCULATE MAXIMIN CONTRASTS
   64.00  128.00  256.00  512.00 1024.00 2048.00
VALUES OF MAXIMIN CONTRAST
 -0.9129 -0.1167 -0.0445  0.0918  0.2791  0.7123

NO. OF MEANS =   5       MAXIMIN R-SQUARED =    0.73803        MAXIMIN R =  0.85909
X'S USED TO CALCULATE MAXIMIN CONTRASTS
  128.00  256.00  512.00 1024.00 2048.00
VALUES OF MAXIMIN CONTRAST
 -0.8944 -0.0726  0.0309  0.2328  0.7032

NO. OF MEANS =   4       MAXIMIN R-SQUARED =    0.79487        MAXIMIN R =  0.89155
X'S USED TO CALCULATE MAXIMIN CONTRASTS
  256.00  512.00 1024.00 2048.00
VALUES OF MAXIMIN CONTRAST
 -0.8660  0.0000  0.1751  0.6910

NO. OF MEANS =   3       MAXIMIN R-SQUARED =    0.87796        MAXIMIN R =  0.93700
X'S USED TO CALCULATE MAXIMIN CONTRASTS
  512.00 1024.00 2048.00
VALUES OF MAXIMIN CONTRAST
 -0.8165  0.1496  [illegible]
```

APPENDIX D

```
SPACING OF X'S IN TREATMENT DESIGN
     0.00      1.00      2.00      4.00      8.00     16.00

NO. OF MEANS =     6        MAXIMIN R-SQUARED =     0.70852            MAXIMIN R =   0.84174
X'S USED TO CALCULATE MAXIMIN CONTRASTS
     0.00      1.00      2.00      4.00      8.00     16.00
VALUES OF MAXIMIN CONTRAST
 -0.9129 -0.0451 -0.0751  0.0674  0.2597  0.7060

NO. OF MEANS =     5        MAXIMIN R-SQUARED =     0.73803            MAXIMIN R =   0.85909
X'S USED TO CALCULATE MAXIMIN CONTRASTS
     1.00      2.00      4.00      8.00     16.00
VALUES OF MAXIMIN CONTRAST
 -0.8944 -0.0726  0.0309  0.2328  0.7032

NO. OF MEANS =     4        MAXIMIN R-SQUARED =     0.79487            MAXIMIN R =   0.89155
X'S USED TO CALCULATE MAXIMIN CONTRASTS
     2.00      4.00      8.00     16.00
VALUES OF MAXIMIN CONTRAST
 -0.8660 -0.0000  0.1751  0.6910

NO. OF MEANS =     3        MAXIMIN R-SQUARED =     0.87796            MAXIMIN R =   0.93700
X'S USED TO CALCULATE MAXIMIN CONTRASTS
     4.00      8.00     16.00
VALUES OF MAXIMIN CONTRAST
 -0.8165  0.1446  0.6719
```